

## A Matrix Approach for General Higher Order Linear Recurrences

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**Abstract.** We consider  $k$  sequences of generalized order- $k$  linear recurrences with arbitrary initial conditions and coefficients, and we give their generalized Binet formulas and generating functions. We also obtain a new matrix method to derive explicit formulas for the sums of terms of the  $k$  sequences. Further, some relationships between determinants of certain Hessenberg matrices and the terms of these sequences are obtained.

2010 Mathematics Subject Classification: 11B37, 40C05, 15A36, 15A15

Keywords and phrases: Higher order recurrence, generating matrix, sum, matrix method.

### 1. Introduction

Linear recurrences have played (and will most certainly play) an important role in many areas of mathematics. Various authors have studied various properties of linear recurrences (such as the well-known Fibonacci and Pell sequences).

In [2], Er defined  $k$  linear recurring sequences of order at most  $k$  as shown: For  $n > 0$  and  $1 \leq i \leq k$ ,

$$g_n^i = \sum_{j=1}^k g_{n-j}^i$$

with initial conditions

$$g_n^i = \begin{cases} 1 & \text{if } n = 1 - i, \\ 0 & \text{otherwise,} \end{cases} \quad \text{for } 1 - k \leq n \leq 0,$$

where  $g_n^i$  is the  $n$ th term of the  $i$ th generalized order- $k$  Fibonacci sequence.

Report Documentation Page				Form Approved OMB No. 0704-0188	
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1. REPORT DATE <b>AUG 2009</b>		2. REPORT TYPE		3. DATES COVERED <b>00-00-2009 to 00-00-2009</b>	
4. TITLE AND SUBTITLE <b>A Matrix Approach for General Higher Order Linear Recurrences</b>				5a. CONTRACT NUMBER	
				5b. GRANT NUMBER	
				5c. PROGRAM ELEMENT NUMBER	
6. AUTHOR(S)				5d. PROJECT NUMBER	
				5e. TASK NUMBER	
				5f. WORK UNIT NUMBER	
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) <b>Naval Postgraduate School, Department of Applied Mathematics, Monterey, CA, 93943</b>				8. PERFORMING ORGANIZATION REPORT NUMBER	
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)				10. SPONSOR/MONITOR'S ACRONYM(S)	
				11. SPONSOR/MONITOR'S REPORT NUMBER(S)	
12. DISTRIBUTION/AVAILABILITY STATEMENT <b>Approved for public release; distribution unlimited</b>					
13. SUPPLEMENTARY NOTES					
14. ABSTRACT <b>We consider k sequences of generalized order-k linear recurrences with arbitrary initial conditions and coefficients, and we give their generalized Binet formulas and generating functions. We also obtain a new matrix method to derive explicit formulas for the sums of terms of the k sequences. Further some relationships between determinants of certain Hessenberg matrices and the terms of these sequences are obtained.</b>					
15. SUBJECT TERMS					
16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT <b>Same as Report (SAR)</b>	18. NUMBER OF PAGES <b>17</b>	19a. NAME OF RESPONSIBLE PERSON
a. REPORT <b>unclassified</b>	b. ABSTRACT <b>unclassified</b>	c. THIS PAGE <b>unclassified</b>			

More generally, in [6], the author gave the generalized order- $k$  Fibonacci and Pell (F-P) sequence as follows: For  $m \geq 0$ ,  $n > 0$  and  $1 \leq i \leq k$

$$u_n^i = 2^m u_{n-1}^i + u_{n-2}^i + \cdots + u_{n-k}^i$$

with initial conditions

$$u_n^i = \begin{cases} 1 & \text{if } n = 1 - i, \\ 0 & \text{otherwise,} \end{cases} \quad \text{for } 1 - k \leq n \leq 0,$$

where  $u_n^i$  is the  $n$ th term of the  $i$ th generalized order- $k$  F-P sequence.

When  $m = 0$ , the generalized order- $k$  F-P sequence  $\{u_n^i\}$  is reduced to the generalized order- $k$  Fibonacci sequence  $\{g_n^i\}$ . Also when  $m = 1$ , the generalized order- $k$  F-P sequence is reduced to the generalized order- $k$  Pell sequence  $\{P_n^i\}$  (for more details see [5]).

Define  $k$  sequences of  $k$ th order linear recurrence relation  $\{f_n^i\}$  as shown, for  $n > 0$  and  $1 \leq i \leq k$

$$(1.1) \quad f_n^i = c_1 f_{n-1}^i + c_2 f_{n-2}^i + \cdots + c_k f_{n-k}^i$$

with initial conditions

$$f_n^i = \begin{cases} 1 & \text{if } n = 1 - i, \\ 0 & \text{otherwise,} \end{cases} \quad \text{for } 1 - k \leq n \leq 0$$

where  $c_j$ ,  $1 \leq j \leq k$ , are real constant coefficients, and  $f_n^i$  is the  $n$ th term of the  $i$ th sequence. When  $k = 2$ ,  $c_1 = c_2 = 1$ , respectively,  $k = c_1 = 2$ ,  $c_2 = 1$  the sequence  $\{f_n^2\}$  is reduced to the Fibonacci sequence  $\{F_n\}$ , respectively, the Pell sequence  $\{P_n\}$ .

Define the  $k \times k$  companion matrix  $A$  and the matrix  $G_n$  as follows:

$$(1.2) \quad A = \begin{bmatrix} c_1 & c_2 & \cdots & c_{k-1} & c_k \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}, \quad G_n = \begin{bmatrix} f_n^1 & f_n^2 & \cdots & f_n^k \\ f_{n-1}^1 & f_{n-1}^2 & \cdots & f_{n-1}^k \\ \vdots & \vdots & \ddots & \vdots \\ f_{n-k+1}^1 & f_{n-k+1}^2 & \cdots & f_{n-k+1}^k \end{bmatrix}$$

Using the approach of Kalman [3], Er [2] showed that

$$(1.3) \quad G_n = A^n$$

and

$$(1.4) \quad f_{n+1}^i = c_i f_n^1 + f_n^{i+1}, \quad \text{for } 1 \leq i \leq k-1$$

$$(1.5) \quad f_{n+1}^k = c_k f_n^1.$$

Matrix methods are helpful and convenient in solving certain problems stemming from linear recursion relations, such as that of finding an explicit expression for the  $n$ th term of the Fibonacci sequence (see [9]), or of analyzing the vibration of a weighted string [10, pp. 152–154]. Here we will consider a more general case using matrix methods to obtain some explicit formulas for the  $n$ th term of a general recurrence relation and the sums of terms of the recurrence. The general linear recurrence relations have been considered by many mathematicians (for references,

see [1, 2, 4, 5]). The authors of [4, 6, 7] give the generalized Binet formula for the generalized order- $k$  Fibonacci, Lucas and Pell numbers by matrix methods.

In this paper, we consider  $k$  sequences of general order- $k$  linear recurrences with  $k$  arbitrary initial conditions and coefficients. Then we study the properties of  $k$  linear recursive sequences and derive many applications to matrices.

## 2. General linear recurrence with $k$ initial conditions

Define a set of  $k$  sequences satisfying the generalized order- $k$  linear recurrence  $\{t_n^i(r_1, r_2, \dots, r_k)\}$  as shown: For  $n > 0$  and  $1 \leq i \leq k$ ,

$$t_n^i = c_1 t_{n-1}^i + c_2 t_{n-1}^i + \dots + c_k t_{n-k}^i$$

with  $k$  initial conditions

$$t_n^i = \begin{cases} r_1 & \text{if } n = 1 - i, \\ r_2 & \text{if } n = 2 - i, \\ \vdots & \vdots \\ r_k & \text{if } n = k - i, \\ 0 & \text{otherwise,} \end{cases} \quad \text{for } 1 - k \leq n \leq 0$$

where the coefficients  $c_i$  and the initial conditions  $r_i$  are arbitrary, for  $1 \leq i \leq k$ , and  $t_n^i$  is the  $n$ th term of  $i$ th sequence. Clearly,  $\{t_n^i(1, 0, \dots, 0)\} = \{f_n^i\}$ , where  $f_n^i$  are given by (1.1).

Next, we define a  $k \times k$  matrix  $H_n = [h_{ij}]$  by

$$(2.1) \quad H_n = \begin{bmatrix} t_n^1 & t_n^2 & \dots & t_n^k \\ t_{n-1}^1 & t_{n-1}^2 & \dots & t_{n-1}^k \\ \vdots & \vdots & \ddots & \vdots \\ t_{n-k+1}^1 & t_{n-k+1}^2 & \dots & t_{n-k+1}^k \end{bmatrix}.$$

By Kalman's [3] approach, we find that

$$(2.2) \quad H_n = AH_{n-1} \text{ and so, } H_n = A^{n-1}H_1,$$

where the matrix  $A$  is given by (1.2).

**Theorem 2.1.** For  $n > 0$ ,

$$t_n^i = \sum_{j=1}^i r_{i+1-j} f_n^j,$$

where  $f_n^i$  is defined as before.

*Proof.* From (2.2), we have  $H_n = A^{n-1}H_1$ . From (2.1) we get

$$H_1 = \begin{bmatrix} t_1^1 & t_1^2 & \dots & t_1^k \\ t_0^1 & t_0^2 & \dots & t_0^k \\ \vdots & \vdots & \ddots & \vdots \\ t_{2-k}^1 & t_{2-k}^2 & \dots & t_{2-k}^k \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^1 c_j r_{2-j} & \sum_{j=1}^2 c_j r_{3-j} & \dots & \sum_{j=1}^k c_j r_{k+1-j} \\ r_1 & r_2 & \dots & r_k \\ 0 & r_1 & \dots & r_{k-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & r_1 \end{bmatrix},$$

which implies that

$$(2.3) \quad H_1 = AE,$$

where the matrix  $E$  is the  $k \times k$  upper tridiagonal matrix of the form

$$E = \begin{bmatrix} r_1 & r_2 & r_3 & \dots & r_k \\ & r_1 & r_2 & \dots & r_{k-1} \\ & & r_1 & \dots & r_{k-2} \\ & & & \ddots & \vdots \\ 0 & & & & r_1 \end{bmatrix}.$$

Using Er's approach [2] and (1.3), we obtain  $A^n = G_n$ . Since  $H_n = A^{n-1}H_1$  and  $H_1 = AE$ , we get

$$(2.4) \quad H_n = A^n E,$$

which can be re-written as

$$(2.5) \quad t_n^i = \sum_{j=1}^i r_{i+1-j} f_n^j,$$

and the proof is complete.  $\blacksquare$

Therefore we see that the general recurrence with arbitrary initial conditions can be written as a linear combination of terms of the recurrence  $\{f_n^i\}$ . By this result, we can easily derive some properties of the recurrence  $\{t_n^i\}$ .

**Corollary 2.1.** *For  $n \in \mathbb{Z}$ ,*

$$\det \begin{pmatrix} t_n^1 & t_n^2 & \dots & t_n^k \\ t_{n-1}^1 & t_{n-1}^2 & \dots & t_{n-1}^k \\ \vdots & \vdots & & \vdots \\ t_{n-k+1}^1 & t_{n-k+1}^2 & \dots & t_{n-k+1}^k \end{pmatrix} = (-1)^{k+1} c_k r_1^k.$$

*Proof.* Let  $H_n, G_n$  and  $E$  be the matrices defined in the proof of Theorem 2.1. It is clear that  $\det G_n = (-1)^{k+1} c_k$  and  $\det E = r_1^k$ . Taking the determinant in  $H_n = G_n E$  shows our claim.  $\blacksquare$

Corollary 2.1 is a vast generalization of the well-known Cassini's identity for the Fibonacci numbers, that is,  $F_n^2 - F_{n-1}F_{n+1} = (-1)^{n-1}$ .

**Corollary 2.2.** *Let  $x^k - c_1x^{k-1} - c_2x^{k-2} - \dots - c_k = (x - \lambda_1) \dots (x - \lambda_k)$  and  $e_n = \lambda_1^n + \lambda_2^n + \dots + \lambda_k^n$ . Then*

$$e_n = \sum_{i=1}^k \left( \sum_{m=1}^i r_{i+1-m} f_{n+1-m}^m \right).$$

*Proof.*  $A$  is the companion matrix from (1.2) and  $x^k - c_1x^{k-1} - c_2x^{k-2} - \dots - c_k$  is its characteristic polynomial, whose roots (also, eigenvalues of  $A$ ) are  $\lambda_1, \dots, \lambda_k$ . Thus the eigenvalues of  $A^n$  are  $\lambda_1^n, \dots, \lambda_k^n$ . Denote the trace of the matrix  $W$  by  $\text{tr}(W)$ . By Theorem 2.1,

$$e_n = \lambda_1^n + \lambda_2^n + \dots + \lambda_k^n = \text{tr}(H_n) = \text{tr}(G_n E)$$

$$= \sum_{i=1}^k \left( \sum_{m=1}^i r_{i+1-m} f_{n+1-t}^m \right).$$

Thus the proof is complete. ■

### 3. Sums of the terms of recurrence $\{t_n^k\}$

In this section we deal with the sums of the terms of recurrence  $\{t_n^k\}$  subscripted from 1 to  $n$ . By the result of Theorem 2.1, clearly

$$(3.1) \quad t_n^k = \sum_{j=1}^k r_{k-j+1} f_n^j.$$

The characteristic polynomial of both the matrix  $A$  and the sequence  $\{f_n^k\}$  is  $E(x) = x^k - c_1 x^{k-1} - c_2 x^{k-2} - \dots - c_{k-1} x - c_k$ . Let  $\lambda_1, \lambda_2, \dots, \lambda_k$  be the characteristic roots of the equation.

**Hypothesis 1.** *Throughout this paper, we suppose that the roots  $\lambda_1, \dots, \lambda_k$  are distinct (which happens if  $\gcd(E, E') = 1$ ) and not equal to 1.*

As special cases, we note that when  $c_i = 1$  for  $1 \leq i \leq k$ , the equation  $x^k - x^{k-1} - \dots - x - 1 = 0$  does not have multiple roots (see [7]). Also, when  $c_1 = 2$  and  $c_i = 1$  for  $2 \leq i \leq k$ , the equation  $x^k - 2x^{k-1} - x^{k-2} - \dots - x - 1 = 0$  does not have multiple roots (see [5]). For the case  $c_1 = 2^m$ ,  $c_i = 1$  for  $2 \leq i \leq k$  and  $m \geq 0$ , we refer to [6].

Let  $V = \Lambda^T$  be a  $k \times k$  Vandermonde matrix, where

$$(3.2) \quad \Lambda = \begin{bmatrix} \lambda_1^{k-1} & \lambda_1^{k-2} & \dots & \lambda_1 & 1 \\ \lambda_2^{k-1} & \lambda_2^{k-2} & \dots & \lambda_2 & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ \lambda_k^{k-1} & \lambda_k^{k-2} & \dots & \lambda_k & 1 \end{bmatrix}.$$

Let  $w_k^i$  be the column matrix

$$w_k^i = \begin{bmatrix} \lambda_1^{n+k-i} \\ \lambda_2^{n+k-i} \\ \vdots \\ \lambda_k^{n+k-i} \end{bmatrix}$$

and  $\Lambda_j^{(i)}$  be the  $k \times k$  matrix obtained from  $\Lambda$  by replacing the  $j$ th column of  $\Lambda$  by  $w_k^i$ .

The generalized Binet formula for the recurrence  $\{f_n^i\}$  can be expressed using  $V = \Lambda^T$  and  $V_j^{(i)} = \Lambda_j^{(i)}$ .

**Theorem 3.1.** *For  $n > 0$  and  $1 \leq i \leq k$ ,*

$$f_{n-i+1}^j = \frac{\det(V_j^{(i)})}{\det(V)}.$$

*Proof.* Since the eigenvalues of  $A$  are distinct (by our Hypothesis 1), we infer that  $A$  is diagonalizable. It is readily seen that  $AV = VD$ , where  $D = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_k)$ . Since  $V$  is invertible,  $V^{-1}AV = D$ . Hence,  $A$  is similar to  $D$ . So we obtain  $A^n V = VD^n$ . Since  $A^n = G_n = [g_{ij}]$ , we obtain the following linear system of equations:

$$\begin{aligned} g_{i1}\lambda_1^{k-1} + g_{i2}\lambda_1^{k-2} + \dots + g_{ik} &= \lambda_1^{n+k-i} \\ g_{i1}\lambda_2^{k-1} + g_{i2}\lambda_2^{k-2} + \dots + g_{ik} &= \lambda_2^{n+k-i} \\ &\vdots \\ g_{i1}\lambda_k^{k-1} + g_{i2}\lambda_k^{k-2} + \dots + g_{ik} &= \lambda_k^{n+k-i}. \end{aligned}$$

Thus, for  $j = 1, 2, \dots, k$ , we get  $g_{ij} = \det(\Lambda_j^{(i)}) / \det(\Lambda)$ , where  $G_n = [g_{ij}]$  and  $g_{ij} = f_{n-i+1}^j$ . The proof is complete.  $\blacksquare$

**Corollary 3.1.** *For  $n > 0$ , we have*

$$t_n^i = \frac{1}{\det(\Lambda)} \sum_{j=1}^i r_{k+1-j} \det(\Lambda_j^{(1)}).$$

For example, when  $c_1 = 2$  and  $c_i = 1$  for all  $2 \leq j \leq k$ , the sequence  $\{f_n^i\}$  is reduced to the generalized order- $k$  Pell sequence  $\{P_n^i\}$  and so the sums of the generalized order- $k$  Pell numbers is given by

$$\sum_{i=1}^n P_i^k = \frac{(P_n^1 + P_n^2 + \dots + P_n^k - 1)}{k}.$$

When  $k = 3$ ,  $c_i = 1$  for  $1 \leq i \leq 3$ , the sequence  $\{f_n^i\}$  is reduced to the generalized Tribonacci sequence  $\{T_n^i\}$  and so

$$\sum_{i=1}^n T_i^3 = \frac{(T_n^1 + T_n^2 + T_n^3 - 1)}{2}$$

and by the definition of the  $\{T_n^i\}$ , we have  $T_n^1 = T_{n+1}^3$  and  $T_n^2 = T_n^3 + T_{n-1}^3$ . For easy writing, we denote  $T_n^3$  by  $T_n$ . Thus we can write

$$\sum_{i=1}^n T_i = \frac{(T_{n+1} + 2T_n + T_{n-1} - 1)}{2} = \frac{(T_{n+2} + T_n - 1)}{2}.$$

We expand our matrix method to find all sums of terms of  $k$  sequences of generalized order- $k$  recurrences  $\{f_n^i\}$  subscripted 1 to  $n$  for all  $1 \leq i \leq k$ .

Define the following two sums: For  $1 \leq i \leq k$ , let  $S_n^{(i)} = \sum_{m=1}^{n-1} f_m^i$  and  $T_n^{(i)} = \sum_{m=1-i}^{n-i} f_m^i$ . Then  $T_n^{(i)} = S_{n-i+1}^{(i)} + 1$ , since

$$f_n^i = \begin{cases} 1 & \text{if } i = 1 - n, \\ 0 & \text{otherwise,} \end{cases} \quad \text{for } 1 - k \leq n \leq 0.$$

Further,

$$(3.3) \quad S_{n+1}^{(i)} = f_n^i + S_n^{(i)}$$

$$(3.4) \quad T_{n+1}^{(i)} = f_{n-i+1}^i + T_n^{(i)}$$

We next define two  $(k+1) \times (k+1)$  matrices as follows:

$$B_i = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & & \\ 0 & & A & \\ 1 & & & \\ 0 & & & \\ \vdots & & & \\ 0 & & & \end{bmatrix} \leftarrow (i+1) \text{ th row}$$

and

$$Y_{n,i} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ S_n^{(i)} & & & \\ S_{n-1}^{(i)} & & & \\ \vdots & & & \\ S_{n-i+2}^{(i)} & & G_n & \\ T_n^{(i)} & & & \\ T_{n-1}^{(i)} & & & \\ \vdots & & & \\ T_{n-k+i}^{(i)} & & & \end{bmatrix} \begin{array}{l} \leftarrow \text{1st row} \\ \leftarrow \text{2nd row} \\ \vdots \\ \leftarrow (i-1) \text{ th row} \\ \leftarrow i \text{ th row} \\ \leftarrow (i+1) \text{ th row} \\ \vdots \\ \leftarrow k \text{ th row} \end{array}$$

where the matrices  $A$  and  $G_n$  were defined before. We have the following result.

**Theorem 3.2.** For  $n > 0$ ,

$$Y_{n,i} = B_i^n.$$

*Proof.* Combining the identities (3.3) and (3.4), we obtain

$$Y_{n+1,i} = Y_{n,i} B_i = \dots = Y_{1,i} B_i^n.$$

From the definitions of  $\{T_n^{(i)}\}$  and  $\{S_n^{(i)}\}$ , we can easily check that  $Y_{1,i} = B_i$ , and the theorem is proven.  $\blacksquare$

Now we are going to derive an explicit expression for every sum  $S_n^{(i)}$  for  $1 \leq i \leq k$  by matrix methods.

We first make some observations. If we expand  $\det B_i$  with respect to the first row, we get

$$\det B_i = \det A$$

and the characteristic polynomials of  $A, B_i$  satisfy

$$C_{B_i}(\lambda) = (1 - \lambda) C_A(\lambda).$$

Since  $\lambda_1, \lambda_2, \dots, \lambda_k$  are the roots of  $C_A(\lambda)$  (distinct and nonequal to 1), the eigenvalues of matrix  $B_i$  are  $\lambda_1, \lambda_2, \dots, \lambda_k, 1$ . Therefore the eigenvalues of the matrix  $B_i$  are distinct, and so  $B_i$  is diagonalizable.



For easy writing, let

$$\mu_i = \frac{\sum_{t=i}^k c_t}{1 - \sum_{t=1}^k c_t} \text{ for } 1 < i \leq k \text{ and } \mu_1 = \frac{1}{1 - \sum_{t=1}^k c_t}.$$

The following  $(k+1) \times (k+1)$  matrix for  $1 < i \leq k$

$$P = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ \mu_i & \lambda_1^{k-1} & \lambda_2^{k-1} & \dots & \lambda_k^{k-1} \\ \mu_i & \lambda_1^{k-2} & \lambda_2^{k-2} & \dots & \lambda_k^{k-2} \\ \vdots & \vdots & \vdots & & \vdots \\ \mu_i & \lambda_1^{k-i+1} & \lambda_2^{k-i+1} & & \lambda_k^{k-i+1} \\ \mu_i + 1 & \lambda_1^{k-i} & \lambda_2^{k-i} & & \lambda_k^{k-i} \\ \mu_i + 1 & \lambda_1^{k-i-1} & \lambda_2^{k-i-1} & & \lambda_k^{k-i-1} \\ \vdots & \lambda_1 & \lambda_2 & \dots & \lambda_k \\ \mu_i + 1 & 1 & 1 & \dots & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ \mu_i & & & & \\ \mu_i & & & & \\ \vdots & & & & \\ \mu_i & & & & \\ \mu_i + 1 & & & & \\ \mu_i + 1 & & & & \\ \vdots & & & & \\ \mu_i + 1 & & & & \end{bmatrix} V$$

satisfies  $B_i P = P D_1$ , where  $D_1$  is the  $(k+1) \times (k+1)$  diagonal matrix defined previously,  $D_1 = \text{diag}(1, \lambda_1, \lambda_2, \dots, \lambda_k)$ . Here we note that if we expand  $\det P$  with respect to the first row, then we get  $\det P = \det \Lambda$ . Since  $\Lambda$  is the Vandermonde matrix, the matrix  $P$  is invertible.

**Theorem 3.3.** For  $n > 0$  and  $1 < i < k$ ,

$$S_n^{(i)} = \mu_i \left( 1 - \sum_{j=1}^k f_n^j \right) - \sum_{m=i}^k f_n^m$$

and

$$S_n^{(1)} = \mu_1 \left( 1 - \sum_{j=1}^k f_n^j \right).$$

*Proof.* Since  $B_i P = P D_1$  for  $1 < i \leq k$  and the matrix  $P$  is invertible, we write  $B_i^n P = P D_1^n$  and so  $Y_{n,i} P = P D_1^n$ . By equating the  $(2, 1)$  entries of the equality  $Y_{n,i} P = P D_1^n$ , we have the conclusion.

For the case  $i = 1$ , one can see that  $B P_1 = P_1 D_1$  where the  $(k+1) \times (k+1)$  matrices  $B$  and  $P_1$  are as follows

$$B = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 1 & & & \\ 0 & & A & \\ \vdots & & & \\ 0 & & & \end{bmatrix} \text{ and } P_1 = \begin{bmatrix} 1 & 0 & \dots & 0 \\ \mu_1 & & & \\ \vdots & & & \\ \mu_1 & & & V \end{bmatrix}.$$

By induction on  $n$ , we see that

$$Y = B^n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ S_n^{(i)} & & & \\ S_{n-1}^{(i)} & & G_n & \\ \vdots & & & \\ S_{n-k+1}^{(i)} & & & \end{bmatrix}.$$

Similar to the cases  $1 < i \leq k$ , the proof is easily seen for the case  $i = 1$ . ■

As a consequence of Theorem 3.3, we get

$$S_n = \sum_{i=1}^n f_i^k = \frac{c_k \left( \sum_{j=1}^k f_n^j - 1 \right)}{c_1 + c_2 + \dots + c_k - 1}.$$

Let  $V_{i,j}$  be a  $k \times k$  matrix obtained from the Vandermonde matrix  $V$  by replacing the  $j$ th column of  $V$  by  $e_i$  where  $V = \Lambda^T$  is defined as in (3.2) and  $e_i$  is the  $i$ th element of the natural basis for  $\mathbb{R}^n$ , that is,

$$e_i = (0, \dots, 0, \underset{\substack{\uparrow \\ \text{ith}}}{1}, 0, \dots, 0)^T$$

and

$$V_{i,j} = \begin{bmatrix} \lambda_1^{k-1} & \dots & \lambda_{j-1}^{k-1} & 0 & \lambda_{j+1}^{k-1} & \dots & \lambda_k^{k-1} \\ \lambda_1^{k-2} & \dots & \lambda_{j-1}^{k-2} & 0 & \lambda_{j+1}^{k-2} & \dots & \lambda_k^{k-2} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \lambda_1^{k-i+1} & \dots & \lambda_{j-1}^{k-i+1} & 0 & \lambda_{j+1}^{k-i+1} & \dots & \lambda_k^{k-i+1} \\ \lambda_1^{k-i} & \dots & \lambda_{j-1}^{k-i} & 1 & \lambda_{j+1}^{k-i} & \dots & \lambda_k^{k-i} \\ \lambda_1^{k-i-1} & \dots & \lambda_{j-1}^{k-i-1} & 0 & \lambda_{j+1}^{k-i-1} & \dots & \lambda_k^{k-i-1} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \lambda_1 & \dots & \lambda_{j-1} & 0 & \lambda_{j+1} & \dots & \lambda_k \\ 1 & \dots & 1 & 0 & 1 & \dots & 1 \end{bmatrix}$$

Let  $q_j^{(i)} = \frac{|V_{i,j}|}{|V|}$ .

**Theorem 3.4.** For any integer  $n$  and  $1 \leq i \leq k$ ,

$$f_n^i = \sum_{j=1}^k q_j^{(i)} \lambda_j^{n+k-1}.$$

*Proof.* We consider the following system of  $k$  linear equations in  $k$  unknowns  $x_1, x_2, \dots, x_k$ :

$$\begin{bmatrix} \lambda_1^{k-1} & \lambda_2^{k-1} & \dots & \lambda_k^{k-1} \\ \vdots & \vdots & & \vdots \\ \lambda_1^{k-i} & \lambda_2^{k-i} & \dots & \lambda_k^{k-i} \\ \vdots & \vdots & & \vdots \\ \lambda_1 & \lambda_2 & \dots & \lambda_k \\ 1 & 1 & \dots & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_j \\ \vdots \\ x_k \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}}_{e_i}.$$

Using Vandermonde's determinants and Cramer rule, we get

$$q_j^{(i)} = \frac{|V_{i,j}|}{|V|} \quad (i = 1, 2, \dots, k),$$

and so, for  $n, k > 0$  and  $1 \leq i \leq k$ ,  $f_n^i = \sum_{j=1}^k q_j^{(i)} \lambda_j^{n+k-1}$ , which completes the proof.  $\blacksquare$

Consequently, we extend the result of Theorem 3.4 to the general order linear recurrences  $\{t_n^i\}$  by the result given by (2.5).

**Corollary 3.2.** *For any integer  $n$  and  $1 \leq i \leq k$ ,*

$$t_n^i = \sum_{j=1}^i \sum_{s=1}^k r_{i+1-j} q_s^{(j)} \lambda_s^{n+k-1}.$$

As an example, we consider the sequence  $\{T_n^i\}$ ,

$$T_n^i = T_{n-1}^i + 3T_{n-2}^i + T_{n-3}^i, n \geq 2, 1 \leq i \leq 3$$

with

$$T_n^i = \begin{cases} 1 & \text{if } i = 1 - n, \\ 0 & \text{otherwise,} \end{cases} \quad \text{for } 1 - k \leq n \leq 0,$$

displayed in the following table:

$i \setminus n$	1	2	3	4	5	6	7	8		
1	1	4	8	21	49	120	288	697	...	$\{T_n^1\}$
2	3	4	13	28	71	168	409	984	...	$\{T_n^2\}$
3	1	1	4	8	21	49	120	288	...	$\{T_n^3\}$

Here we note that  $\gamma_1 = -1$ ,  $\gamma_2 = 1 + \sqrt{2}$ ,  $\gamma_3 = 1 - \sqrt{2}$  and

$$\begin{aligned} q_1^{(1)} &= \frac{1}{(\gamma_1 - \gamma_3)(\gamma_1 - \gamma_2)}, \quad q_2^{(1)} = \frac{1}{(\gamma_2 - \gamma_3)(\gamma_2 - \gamma_1)}, \quad q_3^{(1)} = \frac{1}{(\gamma_2 - \gamma_3)(\gamma_1 - \gamma_3)}, \\ q_1^{(2)} &= -\frac{\gamma_2 + \gamma_3}{(\gamma_1 - \gamma_2)(\gamma_1 - \gamma_3)}, \quad q_2^{(2)} = \frac{\gamma_1 + \gamma_3}{(\gamma_2 - \gamma_3)(\gamma_1 - \gamma_2)}, \quad q_3^{(2)} = -\frac{\gamma_1 + \gamma_2}{(\gamma_2 - \gamma_3)(\gamma_1 - \gamma_3)}, \\ q_1^{(3)} &= \frac{\gamma_2 \gamma_3}{(\gamma_1 - \gamma_3)(\gamma_1 - \gamma_2)}, \quad q_2^{(3)} = -\frac{\gamma_1 \gamma_3}{(\gamma_1 - \gamma_2)(\gamma_2 - \gamma_3)}, \quad q_3^{(3)} = \frac{\gamma_1 \gamma_2}{(\gamma_2 - \gamma_3)(\gamma_1 - \gamma_3)}. \end{aligned}$$

Therefore, by Theorem 3.4, we get

$$\begin{aligned} T_n^1 &= \frac{\gamma_1^{n+2}}{(\gamma_1 - \gamma_3)(\gamma_1 - \gamma_2)} + \frac{\gamma_2^{n+2}}{(\gamma_2 - \gamma_3)(\gamma_2 - \gamma_1)} + \frac{\gamma_3^{n+2}}{(\gamma_2 - \gamma_3)(\gamma_1 - \gamma_3)}, \\ T_n^2 &= -\frac{(\gamma_2 + \gamma_3)\gamma_1^{n+2}}{(\gamma_1 - \gamma_2)(\gamma_1 - \gamma_3)} + \frac{(\gamma_1 + \gamma_3)\gamma_2^{n+2}}{(\gamma_2 - \gamma_3)(\gamma_1 - \gamma_2)} - \frac{(\gamma_1 + \gamma_2)\gamma_3^{n+2}}{(\gamma_2 - \gamma_3)(\gamma_1 - \gamma_3)} \end{aligned}$$

and since  $\gamma_1 \gamma_2 \gamma_3 = 1$ ,

$$\begin{aligned} T_n^3 &= \frac{\gamma_2 \gamma_3 \gamma_1^{n+2}}{(\gamma_1 - \gamma_3)(\gamma_1 - \gamma_2)} - \frac{\gamma_1 \gamma_3 \gamma_2^{n+2}}{(\gamma_1 - \gamma_2)(\gamma_2 - \gamma_3)} + \frac{\gamma_1 \gamma_2 \gamma_3^{n+2}}{(\gamma_2 - \gamma_3)(\gamma_1 - \gamma_3)} \\ &= \frac{\gamma_1^{n+1}}{(\gamma_1 - \gamma_3)(\gamma_1 - \gamma_2)} + \frac{\gamma_2^{n+1}}{(\gamma_2 - \gamma_1)(\gamma_2 - \gamma_3)} + \frac{\gamma_3^{n+1}}{(\gamma_2 - \gamma_3)(\gamma_1 - \gamma_3)} \\ &= T_{n-1}^1. \end{aligned}$$

Observe (from table above) that  $T_n^3 = T_{n-1}^1$ .

#### 4. Generating Functions

In this section we derive the family of generating functions  $G(i, x) = \sum_{n=0}^{\infty} f_n^i x^n$  for the generalized order- $k$  recurrences  $\{f_n^i\}$  for all  $i$ ,  $1 \leq i \leq k$ .

**Theorem 4.1.** For  $1 \leq i \leq k$ ,

$$G(i, x) = \frac{f_0^i + \sum_{m=1}^{k-1} \left( \sum_{v=m+1}^k c_v f_{m-v}^i \right) x^m}{1 - c_1 x - c_2 x^2 - \cdots - c_k x^k}.$$

*Proof.* Let  $G(i, x) = f_0^i x^0 + f_1^i x^1 + f_2^i x^2 + \cdots + f_n^i x^n + \cdots$ . Consider

$$\begin{aligned} & (1 - c_1 x - c_2 x^2 - \cdots - c_k x^k) G(i, x) \\ &= f_0^i + f_1^i x + f_2^i x^2 + \cdots + f_k^i x^k + \cdots + f_n^i x^n + \cdots \\ & \quad - c_1 f_0^i x - c_1 f_1^i x^2 - c_1 f_2^i x^3 - \cdots - c_1 f_{k-1}^i x^k - \cdots - c_1 f_{n-1}^i x^n - \cdots \\ & \quad - c_k f_0^i x^k - c_k f_1^i x^{k+1} - c_k f_2^i x^{k+2} - \cdots - c_k f_{n-k}^i x^n - \cdots \\ &= f_0^i + (f_1^i - c_1 f_0^i) x + (f_2^i - c_1 f_1^i - c_2 f_0^i) x^2 + \cdots \\ & \quad + (f_{k-1}^i - c_1 f_{k-2}^i - c_2 f_{k-3}^i - \cdots - c_{k-1} f_0^i) x^{k-1} \\ & \quad + (f_k^i - c_1 f_{k-1}^i - c_2 f_{k-2}^i - \cdots - c_{k-1} f_0^i - c_k f_1^i) x^k + \cdots \\ & \quad + (f_n^i - c_1 f_{n-1}^i - c_2 f_{n-2}^i - \cdots - c_k f_{n-k}^i) x^n + \cdots. \end{aligned}$$

Now we compute the coefficients of  $x^n$  of the equation above. From the definition of  $\{f_n^i\}$ , we get

$$\begin{aligned} f_1^i &= c_1 f_0^i + c_2 f_{-1}^i + \cdots + c_k f_{1-k}^i \\ &\vdots \\ f_{k-1}^i &= c_1 f_{k-2}^i + c_2 f_{k-3}^i + \cdots + c_{k-1} f_0^i + c_k f_{-1}^i \\ &\vdots \\ f_n^i &= c_1 f_{n-1}^i + c_2 f_{n-2}^i + \cdots + c_k f_{n-k}^i. \end{aligned}$$

and so

$$\begin{aligned} f_1^i - c_1 f_0^i &= c_2 f_{-1}^i + \cdots + c_k f_{1-k}^i \\ f_2^i - c_1 f_1^i - c_2 f_0^i &= c_3 f_{-1}^i + \cdots + c_k f_{2-k}^i \\ &\vdots \\ f_{k-1}^i - c_1 f_{k-2}^i - c_2 f_{k-3}^i - \cdots - c_{k-1} f_0^i &= c_k f_{-1}^i. \end{aligned}$$

Then for  $n \geq k$ , by the definition of  $\{f_n^i\}$ , the coefficients of  $x^n$  are all 0. ■

For example, for fixed  $k$  and  $1 \leq i \leq k$ , we take  $i = 1$ . Thus

$$G(1, x) = f_0^1 x^0 + f_1^1 x^1 + f_2^1 x^2 + \cdots + f_n^1 x^n + \cdots.$$

From the definition of  $\{f_n^i\}$ , the initial conditions of the recurrence  $\{f_n^1\}$  are given by

$$f_n^1 = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{otherwise,} \end{cases} \quad \text{for } 1 - k \leq n \leq 0,$$

which implies

$$(4.1) \quad G(1, x) = \frac{1}{1 - c_1x - c_2x^2 - \dots - c_kx^k}.$$

More generally, we derive the generating function of recurrence  $\{t_n^i\}$ , namely  $g(i, x) = \sum_{k \geq 0} t_k^i x^k$ .

**Corollary 4.1.** *For  $1 \leq i \leq k$ ,*

$$g(i, x) = \frac{t_0^i + \sum_{m=1}^{k-1} \left( \sum_{v=m+1}^k c_v t_{m-v}^i \right) x^m}{1 - c_1x - c_2x^2 - \dots - c_kx^k}.$$

As an example, if we take  $k = i = 2$ ,  $c_1 = c_2 = 1$  and  $r_1 = -1, r_2 = 0$ , then the sequence  $\{t_n^2\}$  is

$$1, 3, 4, 7, 11, 18, 29, \dots$$

which is the well-known Lucas sequence  $\{L_n\}$ . Then by Corollary 4.1, we obtain

$$g(2, x) = \sum_{n=0}^{\infty} t_n^2 x^n = \sum_{n=0}^{\infty} L_n x^n = \frac{t_0^2 - (t_{-1}^2) x^1}{1 - x - x^2}$$

where  $t_0^2 = r_2 = 2$  and  $t_{-1}^2 = r_1 = 1$ . Thus we have the well known result for the Lucas numbers:

$$\sum_{n=0}^{\infty} L_n x^n = \frac{2 - x}{1 - x - x^2}.$$

## 5. $n$ th powers of a companion and $k$ -superdiagonal determinants

In [8], the author gave a relationship between determinants of certain  $n \times n$   $k$ -superdiagonal matrices and the terms of the  $n$ th power of matrix  $A$  given by (1.2). In this section, we derive some new relationships between some Hessenberg determinants and the terms of generalized recurrences  $\{f_n^i\}$  for all  $1 \leq i \leq k$ .

Here, we recall a result of [8]. Define an  $n \times n$   $k$ -superdiagonal matrix  $M_n$  in the following form:

$$M_n = \begin{bmatrix} c_1 & c_2 & \dots & c_k & & 0 \\ -1 & c_1 & c_2 & \dots & c_k & \\ & -1 & c_1 & c_2 & \dots & \ddots \\ & & & \ddots & \dots & \vdots \\ 0 & & & & -1 & c_1 \end{bmatrix}.$$

**Lemma 5.1.** *For  $n > 0$ ,*

$$\det M_n = f_n^1.$$

Indeed, expanding  $\det M_n$  by the elements of the first row gives us

$$(5.1) \quad \det M_n = c_1 \det M_{n-1} + c_2 \det M_{n-2} + \cdots + c_k \det M_{n-k},$$

$$(5.2) \quad = f_n^1 = c_1 f_{n-1}^1 + c_2 f_{n-2}^1 + \cdots + c_k f_{n-k}^1.$$

Now we extend the above result for the generalized sequences  $\{f_n^i\}$  for  $1 \leq i \leq k$ . For this purpose we introduce some new notations: For  $1 \leq t \leq k$ , let  $M_n(t, t+1, \dots, k; r) = [\hat{m}_{ij}]$  denote the matrix obtained from  $M_n = [m_{ij}]$  with  $\hat{m}_{ij} = 0$  for  $i \leq j \leq r$ ,  $i \in \{t, t+1, \dots, k\}$  and otherwise  $\hat{m}_{ij} = m_{ij}$ . Clearly  $M_n(1, 2, \dots, k; 0) = M_n$ .

Recalling that  $G_n = [g_{ij}] = A^n$ , we give the following theorem for the diagonal elements  $g_{jj} = f_{n-j}^{(j+1)}$ .

**Theorem 5.1.** For  $n > j$  and  $1 \leq j \leq k-1$ ,

$$\det M_n(1; j) = f_{n-j}^{j+1}$$

where  $\det M_n(1; 0) = f_n^1$ .

*Proof.* First consider the case  $j = 1$ . If we expand the  $\det M_n(1; 1)$  by the elements of the first row, then

$$\begin{aligned} \det M_n(1; 1) &= 0(\det M_{n-1}) + c_2 \det M_{n-2} + \cdots + c_k \det M_{n-k} \\ &= c_2 \det M_{n-2} + \cdots + c_k \det M_{n-k}. \end{aligned}$$

By (5.1) and (5.2),

$$\begin{aligned} \det M_n(1; 1) &= c_2 f_{n-2}^1 + c_3 f_{n-3}^1 + \cdots + c_k f_{n-k}^1 \\ &= f_n^1 - c_1 f_{n-1}^1 = f_{n-1}^2. \end{aligned}$$

Thus the proof is complete for the case  $j = 1$ .

Now, we take the general case for  $1 \leq j \leq k-1$ . By expanding  $\det M_n(1; j)$  with respect to the first row, we get

$$\det M_n(1; j) = \det \begin{bmatrix} 0 & \cdots & 0 & c_{j+1} & c_{j+2} & \cdots & c_k & 0 & \cdots & 0 \end{bmatrix},$$

which, by (5.1) and (5.2), becomes

$$\begin{aligned} \det M_n(1; j) &= c_{j+1} \det M_{n-j-1} + c_{j+2} \det M_{n-j-2} + \cdots + c_k \det M_{n-k} \\ &= c_{j+1} f_{n-j-1}^1 + c_{j+2} f_{n-j-2}^1 + \cdots + c_k f_{n-k}^1. \end{aligned}$$

From (5.2) and after repeating  $j$  times the identity (1.4), we get

$$\begin{aligned} \det M_n(1; j) &= c_{j+1} f_{n-j-1}^1 + c_{j+2} f_{n-j-2}^1 + \cdots + c_k f_{n-k}^1 \\ &= f_n^1 - c_1 f_{n-1}^1 - c_2 f_{n-2}^1 - \cdots - c_j f_{n-j}^1 \\ &= f_{n-1}^2 - c_2 f_{n-2}^1 - c_3 f_{n-3}^1 - \cdots - c_j f_{n-j}^1 \\ &\quad \dots \\ &= f_{n-j+1}^j - c_j f_{n-j}^1 = f_{n-j}^{j+1}, \end{aligned}$$

and the proof is complete. ■

According to the definition of  $M_n(t, t+1, \dots, k; r)$ , the matrix  $M_n(2, 3; n)$  can be expressed in the compact form

$$M_n(2, 3; n) = \begin{bmatrix} c_1 & c_2 & \dots & c_k & 0 & \dots & \dots & \dots & \dots & 0 \\ -1 & 0 & 0 & \dots & 0 & 0 & \dots & \dots & \dots & 0 \\ & -1 & 0 & 0 & \dots & 0 & 0 & \dots & \dots & 0 \\ & & -1 & c_1 & c_2 & \dots & c_k & 0 & \dots & 0 \\ & & & \ddots & \ddots & \ddots & \dots & \ddots & \ddots & \vdots \\ & & & & -1 & c_1 & c_2 & \dots & c_k & 0 \\ & & & & & -1 & c_1 & c_2 & \dots & c_k \\ & & & & & & \ddots & \ddots & \dots & \vdots \\ & & & & & & & -1 & c_1 & c_2 \\ 0 & & & & & & & & -1 & c_1 \end{bmatrix}.$$

**Theorem 5.2.** For  $n > k + 2$ ,

$$\det M_{n+1}(2, 3, \dots, k; n) = f_{n-k+2}^k.$$

*Proof.* First we consider the case of  $k = 2$ , and  $\det M_{n+1}(2; n)$ . The matrix  $M_n(2; n)$  has the following form:

$$M_n(2; n) = \begin{bmatrix} c_1 & c_2 & \dots & c_k & 0 & \dots & \dots & \dots & 0 \\ -1 & 0 & 0 & \dots & 0 & 0 & \dots & \dots & 0 \\ & -1 & c_1 & c_2 & \dots & c_k & 0 & \dots & 0 \\ & & \ddots & \ddots & \ddots & \dots & \ddots & \ddots & \vdots \\ & & & -1 & c_1 & c_2 & \dots & c_k & 0 \\ & & & & -1 & c_1 & c_2 & \dots & c_k \\ & & & & & \ddots & \ddots & \dots & \vdots \\ & & & & & & -1 & c_1 & c_2 \\ & & & & & & & -1 & c_1 \end{bmatrix}.$$

Expanding  $\det M_{n+1}(2; n)$  with respect to the first row, we obtain

$$\det M_{n+1}(2; n) = c_2 \det M_{n-1} + c_3 \det M_{n-2} + \dots + c_k \det M_{n-k+1}.$$

Since the first principal subdeterminant include a zero row, by Lemma 5.1, we write

$$\begin{aligned} \det M_{n+1}(2; n) &= c_2 f_{n-1}^1 + c_3 f_{n-3}^1 + \dots + c_k f_{n-k+1}^1 \\ &= -c_1 f_n^1 + c_1 f_n^1 + c_2 f_{n-1}^1 + c_3 f_{n-3}^1 + \dots + c_k f_{n-k+1}^1 \\ &= -c_1 f_n^1 + f_{n+1}^1. \end{aligned}$$

By (1.4), we obtain  $\det M_{n+1}(2; n) = -c_1 f_n^1 + f_{n+1}^1 = f_n^2$ . Thus, the proof is complete for  $k = 2$ .

Continuing this expanding process with respect to the first row for the  $\det M_{n+1}(2, 3, \dots, k; n)$ , for  $j \geq 2$ , we get

$$\det M_{n+1}(2, 3, \dots, j; n) = c_j \det M_{n-j+1} + c_{j+1} \det M_{n-j} + \dots + c_k \det M_{n-k+1}$$

which, by Lemma 5.1, gives

$$\begin{aligned}
\det M_{n+1}(2, 3, \dots, j; n) &= c_j f_{n-j+1}^1 + c_3 f_{n-j}^1 + \dots + c_k f_{n-k+1}^1 \\
&= (c_1 f_n^1 + c_2 f_{n-1}^1 + \dots + c_{j-1} f_{n-j+2}^1) \\
&\quad - (c_1 f_n^1 + c_2 f_{n-1}^1 + \dots + c_{j-1} f_{n-j+2}^1) \\
&\quad + c_j f_{n-j+1}^1 + c_3 f_{n-j}^1 + \dots + c_k f_{n-k+1}^1 \\
&= f_{n+1}^1 - (c_1 f_n^1 + c_2 f_{n-1}^1 + \dots + c_{j-1} f_{n-j+2}^1).
\end{aligned}$$

By (1.4), we obtain

$$\begin{aligned}
\det M_{n+1}(2, 3, \dots, j; n) &= f_{n+1}^1 - c_1 f_n^1 - c_2 f_{n-1}^1 - \dots - c_{j-1} f_{n-j+2}^1 \\
&= f_n^2 - c_2 f_{n-1}^1 - \dots - c_{j-1} f_{n-j+2}^1 \\
&\quad \vdots \\
&= f_{n-j+3}^{j-1} - c_{j-1} f_{n-j+2}^1 = f_{n-j+2}^j,
\end{aligned}$$

and the proof is complete. ■

Now we present further relations including other entries of  $G_n$  and the determinant of certain matrices.

Define the  $n \times n$  matrix  $M_n(c_{i,k})$  in the compact form:

$$M_n(c_{i,k}) = \begin{bmatrix} c_i & c_{i+1} & \dots & c_k & 0 & \dots & 0 \\ -1 & & & & & & \\ 0 & & M_{n-1} & & & & \\ \vdots & & & & & & \\ 0 & & & & & & \end{bmatrix}$$

where  $M_n$  is defined as before.

For  $2 \leq t \leq r$ , let  $M_n(c_{i,k}, t, t+1, \dots, r) = [\tilde{m}_{ij}]$  denote the  $n \times n$  matrix obtained from  $M_n(c_{i,k}) = [\tilde{m}_{ij}]$  with taking  $\tilde{m}_{ij} = 0$  for  $i \leq j \leq r$ ,  $i \in \{t, t+1, \dots, n\}$  and otherwise  $\tilde{m}_{ij} = \tilde{m}_{ij}$ .

For example,  $M_7(c_{2,4}, 3, 4)$  takes the form:

$$M_7(c_{2,4}, 3, 4) = \begin{bmatrix} c_2 & c_3 & c_4 & 0 & 0 & 0 & 0 \\ -1 & c_1 & c_2 & c_3 & c_3 & c_4 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & c_1 & c_2 & c_3 \\ 0 & 0 & 0 & 0 & -1 & c_1 & c_2 \\ 0 & 0 & 0 & 0 & 0 & -1 & c_1 \end{bmatrix}.$$

**Theorem 5.3.** For  $n > j - 1$ ,  $2 \leq r \leq k - 1$  and  $2 \leq j \leq k$

$$\det M_n(c_{r,k}, 2, 3, \dots, j) = g_{j-1, j+r-1} = f_{n-j+1}^{j+r-1}$$

where  $G_n = [g_{ij}]$ .



*Proof.* First we prove the case  $r = 2$  and  $2 \leq j \leq k$ . If we expand  $\det M_n(c_{2,k}, 2, 3, \dots, j)$  by the Laplace expansion of determinant, then we obtain the following equation by combining (5.1) and (5.2)

$$\begin{aligned} & \det M_n(c_{2,k}, 2, 3, \dots, j) \\ &= c_{j+1} \det M_{n-j} + c_{j+2} \det M_{n-j-1} + \dots + c_k \det M_{n-k+1} \\ &= c_{j+1} f_{n-j}^1 + c_{j+2} f_{n-j-1}^1 + \dots + c_k f_{n-k+1}^1. \end{aligned}$$

By adding and subtracting  $c_1 f_n^1 + c_2 f_{n-1}^1 + \dots + c_j f_{n-j+1}^1$  to both sides of the above equation, we get

$$\begin{aligned} & \det M_n(c_{2,k}, 2, 3, \dots, j) \\ &= (c_1 f_n^1 + \dots + c_j f_{n-j+1}^1) + c_{j+1} f_{n-j}^1 + \dots + c_k f_{n-k+1}^1 \\ &\quad - (c_1 f_n^1 + c_2 f_{n-1}^1 + \dots + c_j f_{n-j+1}^1) \\ &= f_{n+1}^1 - c_1 f_n^1 - c_2 f_{n-1}^1 - \dots - c_j f_{n-j+1}^1. \end{aligned}$$

By (1.4), we get

$$\begin{aligned} \det M_n(c_{2,k}, 2, 3, \dots, j) &= f_n^2 - c_2 f_{n-1}^1 - c_3 f_{n-2}^1 - \dots - c_j f_{n-j+1}^1 \\ &= f_{n-1}^3 - c_3 f_{n-2}^1 - \dots - c_j f_{n-j+1}^1 \\ &\quad \vdots \\ &= f_{n-j+2}^j - c_j f_{n-j+1}^1 = f_{n-j+1}^{j+1}. \end{aligned}$$

Thus the proof is complete for  $r = 2$ .

Now we consider the case  $r > 2$ . If  $j$  is greater than  $k - 2$ , then the matrix  $M_n(c_{r,k}, 2, 3, \dots, j)$  has a zero row and so we ignore this case. For  $r > 2$  and  $j \leq k - 2$ , we obtain, by (1.4), (5.1) and (5.2)

$$\begin{aligned} & \det M_n(c_{r,k}, 2, 3, \dots, j) \\ &= c_{r+j-1} \det M_{n-j} + \dots + c_k \det M_{n-k+1} \\ &= c_{r+j-1} f_{n-j}^1 + c_{r+j} f_{n-j-1}^1 + \dots + c_k f_{n-k+1}^1 \\ &= f_{n+1}^1 - c_1 f_n^1 - c_2 f_{n-1}^1 - \dots - c_{r+j-2} f_{n-j+1}^1 \\ &= f_n^2 - c_2 f_{n-1}^1 - \dots - c_{r+j-2} f_{n-j+1}^1 \\ &\quad \vdots \\ &= f_{n-j+2}^{r+j-2} - c_{r+j-2} f_{n-j+1}^1 = f_{n-j+1}^{r+j-1}, \end{aligned}$$

which completes the proof for all cases. ■

**Acknowledgement.** The authors would like to thank the referees for constructive comments, which improved the presentation of the paper.

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